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Deterministic inventory model for deteriorating items with capacity constraint and time-proportional backlogging rate

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Abstract

In this paper, a deterministic inventory model for deteriorating items with two warehouses is developed. A rented warehouse is used when the ordering quantity exceeds the limited capacity of the owned warehouse, and it is assumed that deterioration rates of items in the two warehouses may be different. In addition, we allow for shortages in the owned warehouse and assume that the backlogging demand rate is dependent on the duration of the stockout. We obtain the condition when to rent the warehouse and provide simple solution procedures for finding the maximum total profit per unit time. Further, we use a numerical example to illustrate the model and conclude the paper with suggestions for possible future research.

Keywords: Inventory; Deteriorating items; Capacity constraint; Backlogging rate

1. Introduction

The general assumption in classical inventory models is that the organization owns a single warehouse without capacity limitation. In practice, while a large stock is to be held, due to the limited capacity of the owned warehouse (denoted by OW), one additional warehouse is required. This additional warehouse may be a rented warehouse (denoted by RW), which is assumed to be available with abundant capacity. There exist some practical reasons such that the organizations are motivated to order more items than the capacity of OW. For example, the price discount for bulk purchase may be advantageous to the management; the demand of items may be high enough such that a considerable increase in profit is expected; and so on. In these situations, it is generally assumed that the holding cost in RW is higher than that in OW. To reduce the inventory costs, it will be economical to consume the goods of RW at the earliest. As a result, the stocks of OW will not be released until the stocks of RW are exhausted.

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An early discussion on the effect of two warehouses was considered by Hartely [1]. Recently this type of inventory model has been considered by other authors. Sarma [2] developed a deterministic inventory model with infinite replenishment rate and two levels of storage. Murdeshwar and Sathe [3] extended this model to the case of finite replenishment rate. Dave [4] further discussed the cases of bulk release pattern for both finite and infinite replenishment rates. He rectified the errors in Murdeshwar and Sathe [3] and gave a complete solution for the model given by Sarma [2]. In the above literature [2–4], deterioration phenomenon was not taken into account. Assuming the deterioration in both warehouses, Sarma [5] extended his earlier model to the case of infinite replenishment rate with shortages. Pakkala and Achary [6,7] extended the two-warehouse inventory model for deteriorating items with finite replenishment rate and shortages, taking time as discrete and continuous variable, respectively. In these models mentioned above, the demand rate was assumed to be constant. Subsequently, the ideas of time-varying demand and stock-dependent demand were considered by some authors, such as Goswami and Chaudhuri [8,9], Bhunia and Maiti [10,11], Benkherouf [12], Kar et al. [13] and others. In a recent paper, Yang [14] proposed an alternative model for determining the optimal replenishment cycle for the two-warehouse inventory problem under inflation, in which the inventory deteriorates at a constant rate over times and shortages were allowed. She then proved that the optimal solution not only exists but also is unique.

Furthermore, the characteristics of all above papers are that shortages are not allowed or assumed to be completely backlogged. Zhou [15] presented a multi-warehouse inventory model for non-perishable items with time-varying demand and partial backlogging. In his model, the backlogging function was assumed to be dependent on the amount of demand backlogged. In many cases customers are conditioned to a shipping delay, and may be willing to wait for a short time in order to get their first choice. Generally speaking, the length of the waiting time for the next replenishment is the main factor for deciding whether the backlogging will be accepted or not. The willingness of a customer to wait for backlogging during a shortage period declines with the length of the waiting time. Therefore, a situation is quite likely to arise in which that many savvy retailers suggest replacement items, and also provide the restocking date to allow the customer to wait during the stockout period. To reflect this phenomenon, Abad [16,17] discussed a pricing and lot-sizing problem for a product with a variable rate of deterioration, allowing shortages and partial backlogging. The backlogging rate depends on the time to replenishment—the longer customers must wait, the greater the fraction of lost sales. However, he does not use the stockout cost (includes backorder cost and the lost sale cost) in the formulation of the objective function since these costs are not easy to estimate, and its immediate impact is that there is a lower service level to customers.

Companies have recognized that besides maximizing profit, customer satisfaction plays an important role for getting and keeping a successful position in a competitive market. The proper inventory level should be set based on the relationship between the investment in inventory and the service level. With a lost sale, the customer's demand for the item is lost and presumably filled by a competitor. It can be considered as the loss of profit on the sales. Moreover, it also includes the cost of losing the customer, loss of goodwill, and of establishing a poor record of service. Therefore, if we omit the stockout cost from the total profit, then the profit will be overrated. It is true that the stockout cost is very difficult to measure. However, this does not mean that the unit does not have some specific values. In practice, the stockout cost can be easy to obtain from accounting data. In this paper, we develop a deterministic inventory model for deteriorating items with two warehouses. We assume that the inventory costs (including holding cost and deterioration cost) in RW are higher than those in OW. In addition, shortages are allowed in the owned warehouse and the backlogging rate of unsatisfied demand is assumed to be a decreasing function of the waiting time. We then prove that the optimal replenishment policy not only exists but also is unique. Moreover, a numerical example is used to illustrate the proposed model, and concluding remarks are provided.

2. Notation and assumptions

2.1. Notation

To develop the mathematical model of inventory replenishment schedule with two warehouses, the notation adopted in this paper is as below:

D	the demand rate per unit time
A	the replenishment cost per order
C	the purchasing cost per unit
S	the selling price per unit, where $S > C$
W	the capacity of the owned warehouse
Q	the ordering quantity per cycle
B	the maximum inventory level per cycle
C_{11}	the holding cost per unit per unit time in OW
C_{12}	the holding cost per unit per unit time in RW, where $C_{12} > C_{11}$
C_2	the shortage cost per unit per unit time
R	the opportunity cost (i.e., goodwill cost) per unit
α	the deterioration rate in OW, where $0 \leqslant \alpha < 1$
β	the deterioration rate in RW, where $0 \leqslant \beta < 1$
t_w	the time at which the inventory level reaches zero in RW
t_1	the time at which the inventory level reaches zero in OW
t_2	the length of period during which shortages are allowed
T	the length of the inventory cycle, hence $T = t_1 + t_2$
$I_1(t)$	the level of positive inventory in RW at time t
$I_2(t)$	the level of positive inventory in OW at time t
$I_3(t)$	the level of negative inventory at time t
$P(t_w, t_2)$	the total profit per unit time in the two-warehouse case
$\Pi(t_1, t_2)$	the total profit per unit time under the case without capacity constraint in OW

2.2. Assumptions

In addition, the following assumptions are imposed:

1. Replenishment rate is infinite, and lead time is zero.
2. The time horizon of the inventory system is infinite.
3. The owned warehouse (OW) has a fixed capacity of W units; the rented warehouse (RW) has unlimited capacity.
4. The goods of OW are consumed only after consuming the goods kept in RW.
5. To guarantee the optimal solution exists, we assume that the maximum deteriorating quantity for items in OW, αW, is less than the demand rate D; that is, $\alpha W < D$.
6. The unit inventory costs (including holding cost and deterioration cost) per unit time in RW are higher than those in OW; that is, $C_{12} + \beta C > C_{11} + \alpha C$.
7. Shortages are allowed. Unsatisfied demand is backlogged, and the fraction of shortages backordered is $\frac{1}{1+\delta x}$, where x is the waiting time up to the next replenishment and δ is a positive constant.

3. Mathematical formulation

Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, we consider the following time intervals separately, $[0, t_w]$, $[t_w, t_1]$, and $[t_1, T]$. During the interval $[0, t_w]$, the inventory levels are positive at RW and OW. At RW, the inventory is depleted due to the combined effects of demand and deterioration. At OW, the inventory is only depleted by the effect of deterioration. Hence, the inventory level at RW and OW are governed by the following differential equations:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -D - \beta I_1(t), \quad 0 < t < t_w, \tag{1}$$

with the boundary condition $I_1(t_w) = 0$ and

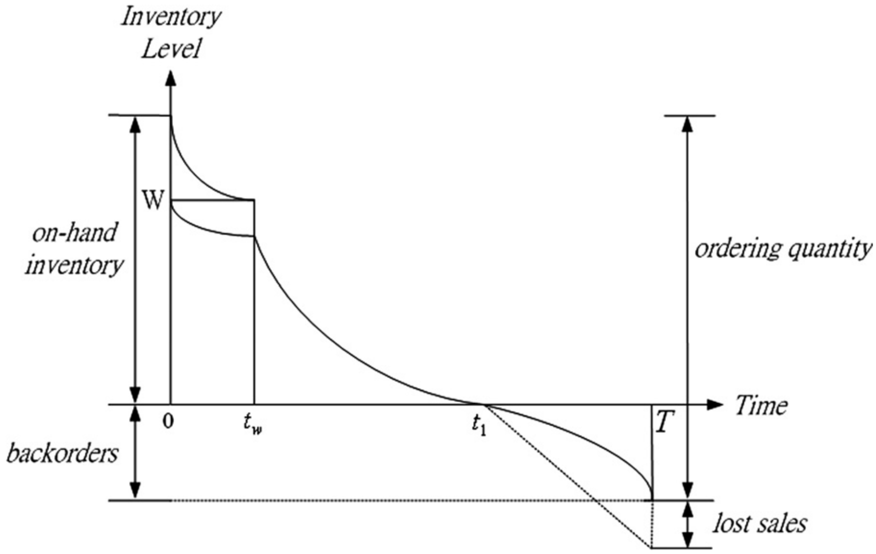

Fig. 1. Graphical representation of a two-warehouse inventory system.

$$\frac{dI_2(t)}{dt} = -\alpha I_2(t), \quad 0 < t < t_w, \tag{2}$$

with the initial condition $I_2(0) = W$, respectively. Solving the differential equations (1) and (2) respectively, we get the inventory level as follows:

$$I_1(t) = \frac{D}{\beta}[e^{\beta(t_w - t)} - 1], \quad 0 \leqslant t \leqslant t_w \tag{3}$$

and

$$I_2(t) = We^{-\alpha t}, \quad 0 \leqslant t \leqslant t_w. \tag{4}$$

During the interval $[t_w, t_1]$, the inventory in OW is depleted due to the combined effects of demand and deterioration. Hence, the inventory level at OW is governed by the following differential equation:

$$\frac{dI_2(t)}{dt} = -D - \alpha I_2(t), \quad t_w < t < t_1, \tag{5}$$

with the boundary condition $I_2(t_1) = 0$. Solving the differential equation (5), we obtain the inventory level as

$$I_2(t) = \frac{D}{\alpha}[e^{\alpha(t_1 - t)} - 1], \quad t_w \leqslant t \leqslant t_1. \tag{6}$$

Due to continuity of $I_2(t)$ at point $t = t_w$, it follows from Eqs. (4) and (6), then

$$We^{-\alpha t_w} = \frac{D}{\alpha}[e^{\alpha(t_1 - t_w)} - 1], \tag{7}$$

which implies that

$$t_1 = t_w + \frac{1}{\alpha}\ln\left(1 + \frac{\alpha W e^{-\alpha t_w}}{D}\right). \tag{8}$$

It notes that t_1 is a function of t_w. Then taking the first-order derivative of t_1 with respect to t_w, it yields

$$\frac{dt_1}{dt_w} = \frac{1}{1 + \alpha W e^{-\alpha t_w}/D} < 1. \tag{9}$$

Thus $\frac{dt_1}{dt_w} - 1 < 0$ holds.

Furthermore, at time t_1, the inventory level reaches zero in OW and shortage occurs. During $[t_1, T]$, the inventory level only depend on demand, and some demand is lost while a fraction $\frac{1}{1+\delta(T-t)}$ of the demand is backlogged, where $t \in [t_1, T]$. The inventory level is governed by the following differential equation:

$$\frac{\mathrm{d}I_3(t)}{\mathrm{d}t} = -\frac{D}{1+\delta(T-t)}, \quad t_1 < t < T, \tag{10}$$

with the boundary condition $I_3(t_1) = 0$. Solving the differential equation (10), we obtain the inventory level as

$$I_3(t) = -\frac{D}{\delta}\{\ln[1+\delta(T-t_1)] - \ln[1+\delta(T-t)]\}, \quad t_1 \leqslant t \leqslant T. \tag{11}$$

Therefore, the ordering quantity over the replenishment cycle can be determined as

$$Q = I_1(0) + I_2(0) - I_3(t) = \frac{D(\mathrm{e}^{\beta t_\mathrm{w}} - 1)}{\beta} + W + \frac{D\ln(1+\delta t_2)}{\delta} \tag{12}$$

and the maximum inventory level per cycle is

$$B = I_1(0) + I_2(0) = \frac{D(\mathrm{e}^{\beta t_\mathrm{w}} - 1)}{\beta} + W. \tag{13}$$

Based on Eqs. (3), (4), (6) and (11), the total profit per cycle consists of the following elements:

1. ordering cost per cycle $= A$,
2. holding cost per cycle in RW

$$\begin{aligned} &= C_{12}\int_0^{t_\mathrm{w}} I_1(t)\,\mathrm{d}t \\ &= C_{12}D(\mathrm{e}^{\beta t_\mathrm{w}} - \beta t_\mathrm{w} - 1)/\beta^2, \end{aligned}$$

3. holding cost per cycle in OW

$$\begin{aligned} &= C_{11}\left(\int_0^{t_\mathrm{w}} I_2(t)\,\mathrm{d}t + \int_{t_\mathrm{w}}^{t_1} I_2(t)\,\mathrm{d}t\right) \\ &= C_{11}\{W(1-\mathrm{e}^{-\alpha t_\mathrm{w}})/\alpha + D[\mathrm{e}^{\alpha(t_1-t_\mathrm{w})} - 1 - \alpha(t_1-t_\mathrm{w})]/\alpha^2\} \\ &= C_{11}[W - D(t_1-t_\mathrm{w})]/\alpha \quad \text{(by Eq. (7))}, \end{aligned}$$

4. shortage cost per cycle

$$\begin{aligned} &= C_2\int_{t_1}^{T} -I_3(t)\,\mathrm{d}t \\ &= C_2D\{\delta(T-t_1) - \ln[1+\delta(T-t_1)]\}/\delta^2 \\ &= C_2D[\delta t_2 - \ln(1+\delta t_2)]/\delta^2, \quad \text{where } t_2 = T - t_1, \end{aligned}$$

5. opportunity cost due to lost sales per cycle

$$\begin{aligned} &= RD\int_{t_1}^{T}\{1 - 1/[1+\delta(T-t)]\}\,\mathrm{d}t \\ &= RD\{\delta(T-t_1) - \ln[1+\delta(T-t_1)]\}/\delta \\ &= RD[\delta t_2 - \ln(1+\delta t_2)]/\delta, \end{aligned}$$

6. purchase cost per cycle

$$\begin{aligned} &= CQ \\ &= C\{W + D(\mathrm{e}^{\beta t_\mathrm{w}} - 1)/\beta + D\ln[1+\delta(T-t_1)]/\delta\} \\ &= C[W + D(\mathrm{e}^{\beta t_\mathrm{w}} - 1)/\beta + D\ln(1+\delta t_2)/\delta], \end{aligned}$$

7. sales revenue per cycle

$$
\begin{aligned}
&= S\left\{\int_0^{t_1} D\,\mathrm{d}t + \int_{t_1}^{T} D/[1+\delta(T-t)]\,\mathrm{d}t\right\} \\
&= SD\{\delta t_1 + \ln[1+\delta(T-t_1)]\}/\delta \\
&= SD[\delta t_1 + \ln(1+\delta t_2)]/\delta.
\end{aligned}
$$

Therefore, the total profit per unit time of our model is obtained as follows:

$$
\begin{aligned}
P(t_\mathrm{w}, t_2) &= \frac{1}{t_1+t_2}\{\text{sales revenue} - \text{ordering cost} - \text{holding cost} - \text{shortage cost} \\
&\quad - \text{opportunity cost} - \text{purchase cost}\} \\
&= D(S-C) - \frac{A}{t_1+t_2} - \frac{C}{t_1+t_2}\left[W + \frac{D}{\beta}(\mathrm{e}^{\beta t_\mathrm{w}}-1) - Dt_1\right] - \frac{C_{11}}{\alpha(t_1+t_2)}[W - D(t_1-t_\mathrm{w})] \\
&\quad - \frac{DC_{12}}{\beta^2(t_1+t_2)}(\mathrm{e}^{\beta t_\mathrm{w}} - \beta t_\mathrm{w} - 1) - \frac{D[C_2+\delta(S-C+R)]}{\delta^2(t_1+t_2)}[\delta t_2 - \ln(1+\delta t_2)],
\end{aligned} \tag{14}
$$

where t_1 is a function of t_w and is defined as in Eq. (8).

To maximize the total profit per unit time, taking the first derivative of $P(t_\mathrm{w}, t_2)$ with respect to t_w and t_2, respectively, we obtain

$$
\begin{aligned}
\frac{\partial P(t_\mathrm{w},t_2)}{\partial t_\mathrm{w}} &= \frac{D(S-C) - P(t_\mathrm{w},t_2)}{t_1+t_2}\frac{\mathrm{d}t_1}{\mathrm{d}t_\mathrm{w}} - \frac{D}{t_1+t_2}\left[C\left(\mathrm{e}^{\beta t_\mathrm{w}} - \frac{\mathrm{d}t_1}{\mathrm{d}t_\mathrm{w}}\right) - \frac{C_{11}}{\alpha}\left(\frac{\mathrm{d}t_1}{\mathrm{d}t_\mathrm{w}} - 1\right) + \frac{C_{12}}{\beta}(\mathrm{e}^{\beta t_\mathrm{w}}-1)\right] \\
&= \frac{1}{(t_1+t_2)(1+\alpha W\mathrm{e}^{-\alpha t_\mathrm{w}}/D)}\left\{D(S-C) - P(t_\mathrm{w},t_2) - D\left[(C_{11}+\alpha C)\frac{W\mathrm{e}^{-\alpha t_\mathrm{w}}}{D}\right.\right. \\
&\quad \left.\left. + \frac{(C_{12}+\beta C)(\mathrm{e}^{\beta t_\mathrm{w}}-1)}{\beta}\left(1 + \frac{\alpha W\mathrm{e}^{-\alpha t_\mathrm{w}}}{D}\right)\right]\right\}
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
\frac{\partial P(t_\mathrm{w},t_2)}{\partial t_2} &= \frac{1}{(t_1+t_2)^2}\left\{A + C\left[W + \frac{D}{\beta}(\mathrm{e}^{\beta t_\mathrm{w}}-1) - Dt_1\right] + \frac{C_{11}}{\alpha}[W - D(t_1-t_\mathrm{w})] + \frac{DC_{12}}{\beta^2}(\mathrm{e}^{\beta t_\mathrm{w}} - \beta t_\mathrm{w} - 1)\right. \\
&\quad \left. + \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)] - \frac{D[C_2+\delta(S-C+R)](t_1+t_2)t_2}{1+\delta t_2}\right\} \\
&= \frac{1}{t_1+t_2}\left\{D(S-C) - P(t_\mathrm{w},t_2) - \frac{D[C_2+\delta(S-C+R)]t_2}{1+\delta t_2}\right\}.
\end{aligned} \tag{16}
$$

The optimal solution of (t_w, t_2) must satisfy the equations $\frac{\partial P(t_\mathrm{w},t_2)}{\partial t_\mathrm{w}} = 0$ and $\frac{\partial P(t_\mathrm{w},t_2)}{\partial t_2} = 0$, simultaneously. Solving these two equations, we obtain

$$
D(S-C) - P(t_\mathrm{w},t_2) = D\left[(C_{11}+\alpha C)\frac{W\mathrm{e}^{-\alpha t_\mathrm{w}}}{D} + \frac{(C_{12}+\beta C)(\mathrm{e}^{\beta t_\mathrm{w}}-1)}{\beta}\left(1 + \frac{\alpha W\mathrm{e}^{-\alpha t_\mathrm{w}}}{D}\right)\right] \tag{17}
$$

and

$$
D(S-C) - P(t_\mathrm{w},t_2) = \frac{D[C_2+\delta(S-C+R)]t_2}{1+\delta t_2}, \tag{18}
$$

respectively. Because both the left hand sides in Eqs. (17) and (18) are the same, the right hand sides in these equations are equal, that is,

$$
\frac{[C_2+\delta(S-C+R)]t_2}{1+\delta t_2} = (C_{11}+\alpha C)\frac{W\mathrm{e}^{-\alpha t_\mathrm{w}}}{D} + \frac{(C_{12}+\beta C)(\mathrm{e}^{\beta t_\mathrm{w}}-1)}{\beta}\left(1 + \frac{\alpha W\mathrm{e}^{-\alpha t_\mathrm{w}}}{D}\right). \tag{19}
$$

Furthermore, we substitute $P(t_w, t_2)$ in (14) into Eq. (18) and obtain

$$\frac{D[C_2+\delta(S-C+R)](t_1+t_2)t_2}{1+\delta t_2} = A + C\left[W + \frac{D}{\beta}(e^{\beta t_w}-1) - Dt_1\right] + \frac{C_{11}}{\alpha}[W - D(t_1-t_w)] + \frac{DC_{12}}{\beta^2}(e^{\beta t_w}-\beta t_w - 1) + \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)]. \tag{20}$$

Now, we let $K(t_w)$ denote the right hand side of Eq. (19), that is,

$$K(t_w) = (C_{11}+\alpha C)\frac{We^{-\alpha t_w}}{D} + \frac{(C_{12}+\beta C)(e^{\beta t_w}-1)}{\beta}\left(1+\frac{\alpha We^{-\alpha t_w}}{D}\right), \quad t_w \geqslant 0. \tag{21}$$

Then we have:

Lemma 1. *$K(t_w)$ is a continuous and strictly increasing function of $t_w \in [0,\infty)$, and its range is $[W(C_{11}+\alpha C)/D, \infty)$.*

Proof. Taking the derivative of $K(t_w)$ with respect to t_w, we have

$$\begin{aligned}\frac{dK(t_w)}{dt_w} &= (C_{12}+\beta C)e^{\beta t_w}\left(1+\frac{\alpha We^{-\alpha t_w}}{D}\right) - \frac{\alpha}{\beta}(C_{12}+\beta C)(e^{\beta t_w}-1)\frac{\alpha We^{-\alpha t_w}}{D} - (C_{11}+\alpha C)\frac{\alpha We^{-\alpha t_w}}{D}\\ &= (C_{12}+\beta C)\frac{\alpha We^{-\alpha t_w}}{D}H(t_w) + [(C_{12}+\beta C)-(C_{11}+\alpha C)]\frac{\alpha We^{-\alpha t_w}}{D},\end{aligned}$$

where

$$H(t_w) = \frac{D}{W}\frac{e^{(\alpha+\beta)t_w}}{\alpha} + \frac{(e^{\beta t_w}-1)}{\beta}(\beta-\alpha), \quad t_w \geqslant 0.$$

Because

$$\frac{dH(t_w)}{dt_w} = \frac{D}{W}\frac{\alpha+\beta}{\alpha}e^{(\alpha+\beta)t_w} + e^{\beta t_w}(\beta-\alpha) > e^{\beta t_w}\left[\frac{D}{\alpha W}(\alpha+\beta) - (\alpha+\beta)\right] > (\alpha+\beta)\left(\frac{D}{\alpha W}-1\right) > 0$$
(by Assumption 5),

$H(t_w)$ is a strictly increasing function of $t_w \in [0,\infty)$, which implies

$$H(t_w) > H(0) = \frac{D}{\alpha W} > 0, \quad \text{for } t_w > 0.$$

Thus, from the above result and Assumption 6, we know that $\frac{dK(t_w)}{dt_w} > 0$, for $t_w > 0$. Therefore, $K(t_w)$ is a strictly increasing function of $t_w \in [0,\infty)$. The fact that $K(0) = W(C_{11}+\alpha C)/D$ and $\lim_{t_w\to\infty} K(t_w) = \infty$ are obvious. This completes the proof. □

For any given $t_w \in [0,\infty)$, from Eq. (19), we define a function

$$F(t_2) = [C_2+\delta(S-C+R)]\frac{t_2}{1+\delta t_2} - K(t_w), \quad t_2 \geqslant 0, \tag{22}$$

then, if $\frac{C_2+\delta(S-C+R)}{\delta} \leqslant K(0) = \frac{W(C_{11}+\alpha C)}{D}$, we have

$$F(t_2) < \frac{C_2+\delta(S-C+R)}{\delta} - K(t_w) < \frac{C_2+\delta(S-C+R)}{\delta} - K(0) = \frac{C_2+\delta(S-C+R)}{\delta} - \frac{W(C_{11}+\alpha C)}{D} \leqslant 0, \quad \text{for } t_2 \in [0,\infty),$$

which implies, for any given $t_w \in [0,\infty)$, there does not exist a value $t_2 \in [0,\infty)$ such that $F(t_2) = 0$, i.e., we can not find a value t_2 which satisfies Eq. (19). However, for this situation, from Eq. (15), we have

$$\frac{\partial P(t_w,t_2)}{\partial t_w} = \frac{D}{t_1+t_2}\frac{1}{1+\frac{\alpha We^{-\alpha t_w}}{D}}F(t_2) < 0.$$

Thus, when $\frac{C_2+\delta(S-C+R)}{\delta} \leqslant \frac{W(C_{11}+\alpha C)}{D}$ or equivalently, $W \geqslant \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, the maximum value of $P(t_w, t_2)$ occurs at the boundary point $t_w^* = 0$.

In the special circumstance that $t_w^* = 0$, the optimal value of t_1 (denoted by t_1^*) can be obtained by Eq. (8) and is $t_1^* = \frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)$. Besides, from Eq. (13), the maximum inventory level per cycle is $B^* = W$. Then, plunging t_w^* and t_1^* into Eq. (20), the optimal value of t_2 should be selected to satisfy

$$D[C_2+\delta(S-C+R)]\frac{(t_1^*+t_2)t_2}{1+\delta t_2} = A + \left(C+\frac{C_{11}}{\alpha}\right)(W-Dt_1^*) + \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)]. \tag{23}$$

Now, we want to prove that the value of t_2 which satisfies Eq. (23) is unique. Let

$$Z(t_2) = \frac{D[C_2+\delta(S-C+R)](t_1^*+t_2)t_2}{1+\delta t_2} - A - \left(C+\frac{C_{11}}{\alpha}\right)(W-Dt_1^*) - \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)], \quad \text{for } t_2 \geqslant 0.$$

The derivative of $Z(t_2)$ with respect to t_2 is

$$\frac{\mathrm{d}Z(t_2)}{\mathrm{d}t_2} = \frac{D[C_2+\delta(S-C+R)](t_1^*+t_2)}{(1+\delta t_2)^2} > 0,$$

thus, $Z(t_2)$ is a strictly increasing function of $t_2 \in [0,\infty)$. Furthermore, we have $Z(0) = -A - \left(C+\frac{C_{11}}{\alpha}\right)(W-Dt_1^*) < 0$, and $\lim_{t_2\to\infty} Z(t_2) = \infty$. By using the Intermediate Value Theorem, there exists a unique solution $t_2 = t_2^* \in (0,\infty)$ such that $Z(t_2^*) = 0$, that is, t_2^* is the unique value which satisfies Eq. (23). Summarize the above arguments, we obtain the following theorem.

Theorem 1. *If* $W \geqslant \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, *then the optimal value of* (t_w, t_1, t_2) *is given by* $t_w^* = 0$, $t_1^* = \frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)$, *and* t_2^* *is the value which satisfies Eq. (23).*

Theorem 1 shows that if $W \geqslant \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, then the capacity of the OW is sufficient and the maximum inventory level per cycle is $B^* = W$. Besides, the optimal inventory cycle is $T^* = t_1^* + t_2^*$. Once the optimal solution $(t_w^*, t_2^*) = (0, t_2^*)$ is obtained, we substitute $(0, t_2^*)$ into Eqs. (12) and (14) together with $t_1^* = \frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)$, the optimal ordering quantity per cycle (denoted by Q^*) and the maximum total profit per unit time $P(0, t_2^*)$ are as follows:

$$Q^* = W + \frac{D\ln(1+\delta t_2^*)}{\delta}$$

and

$$\begin{aligned} P(0,t_2^*) &= D(S-C) - \frac{1}{t_1^*+t_2^*}\left\{A + \left(C+\frac{C_{11}}{\alpha}\right)(W-Dt_1^*) + \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2^* - \ln(1+\delta t_2^*)]\right\} \\ &= D(S-C) - \frac{D[C_2+\delta(S-C+R)]t_2^*}{(1+\delta t_2^*)}. \end{aligned} \tag{24}$$

Next, we consider the case: $\frac{C_2+\delta(S-C+R)}{\delta} > K(0) = \frac{W(C_{11}+\alpha C)}{D}$. From Lemma 1, $K(t_w)$ is a continuous and strictly increasing function of $t_w \in [0,\infty)$, thus we can find a unique value $\hat{t}_w \in (0,\infty)$ such that $K(\hat{t}_w) = \frac{C_2+\delta(S-C+R)}{\delta}$. Furthermore, for $t_w \geqslant \hat{t}_w$, we have

$$\begin{aligned} K(t_w) \geqslant K(\hat{t}_w) &= \frac{C_2+\delta(S-C+R)}{\delta} > \frac{C_2+\delta(S-C+R)}{\delta} - \frac{C_2+\delta(S-C+R)}{\delta}\frac{1}{1+\delta t_2} \\ &= \frac{[C_2+\delta(S-C+R)]t_2}{1+\delta t_2}. \end{aligned}$$

It implies that Eq. (19) does not hold for $t_w \in [\hat{t}_w, \infty)$. Therefore, the optimal solution of t_w which satisfies Eq. (19) will occur in the interval $(0, \hat{t}_w)$.

On the other hand, from the definition of $F(t_2)$ in Eq. (22), it can be shown that $F(t_2)$ is a continuous and strictly increasing function of $t_2 \in [0, \infty)$. Besides, we have $F(0) = -K(t_w) < -K(0) = -\frac{W(C_{11}+\alpha C)}{D} < 0$, and for any given $t_w \in (0, \hat{t}_w)$,

$$\lim_{t_2 \to \infty} F(t_2) = \frac{C_2 + \delta(S - C + R)}{\delta} - K(t_w) > \frac{C_2 + \delta(S - C + R)}{\delta} - K(\hat{t}_w) = 0.$$

Thus, there exists a unique value $t_2 \in (0, \infty)$ such that $F(t_2) = 0$. Consequently, when $\frac{C_2+\delta(S-C+R)}{\delta} > \frac{W(C_{11}+\alpha C)}{D}$, or equivalent, $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, and for any given $t_w \in (0, \hat{t}_w)$, we can find a unique value $t_2 \in (0, \infty)$ such that

$$[C_2 + \delta(S - C + R)] \frac{t_2}{1 + \delta t_2} = K(t_w). \tag{25}$$

From Eq. (25), we obtain

$$t_2 = \frac{K(t_w)}{C_2 + \delta(S - C + R) - \delta K(t_w)}. \tag{26}$$

Thus, t_2 is a function of $t_w \in (0, \hat{t}_w)$, and further we have

$$\frac{dt_2}{dt_w} = \frac{[C_2 + \delta(S - C + R)] \frac{dK(t_w)}{dt_w}}{[C_2 + \delta(S - C + R) - \delta K(t_w)]^2} > 0. \tag{27}$$

Once the value $t_w^* \in (0, \hat{t}_w)$ is obtained, the optimal solutions of t_1, t_2 and T (denoted by t_1^*, t_2^* and T^*, respectively) are as follows

$$t_1^* = t_w^* + \frac{1}{\alpha} \ln\left(1 + \frac{\alpha W e^{-\alpha t_w^*}}{D}\right), \tag{28}$$

$$t_2^* = \frac{K(t_w^*)}{C_2 + \delta(S - C + R) - \delta K(t_w^*)} \tag{29}$$

and

$$T^* = t_1^* + t_2^*. \tag{30}$$

Now, we want to prove the existence of t_w^* in $(0, \hat{t}_w)$. Motivated by Eq. (20), we let

$$\begin{aligned} G(t_w) = {} & A + C\left[W + \frac{D}{\beta}(e^{\beta t_w} - 1) - Dt_1\right] + \frac{C_{11}}{\alpha}[W - D(t_1 - t_w)] + \frac{DC_{12}}{\beta^2}(e^{\beta t_w} - \beta t_w - 1) \\ & + \frac{D[C_2 + \delta(S - C + R)]}{\delta^2}[\delta t_2 - \ln(1 + \delta t_2)] - D[C_2 + \delta(S - C + R)]\frac{(t_1 + t_2)t_2}{1 + \delta t_2}, \quad t_w \in [0, \hat{t}_w), \end{aligned} \tag{31}$$

where t_1 and t_2 are defined as in Eqs. (8) and (26), respectively. Due to the relations shown in Eqs. (19) and (27), the derivative of $G(t_w)$ with respect to $t_w \in (0, \hat{t}_w)$ yields

$$\begin{aligned} \frac{dG(t_w)}{dt_w} &= DC\left(e^{\beta t_w} - \frac{dt_1}{dt_w}\right) - \frac{DC_{11}}{\alpha}\left(\frac{dt_1}{dt_w} - 1\right) + \frac{DC_{12}}{\beta}(e^{\beta t_w} - 1) \\ &\quad - D[C_2 + \delta(S - C + R)]\frac{t_1 + t_2}{(1 + \delta t_2)^2}\frac{dt_2}{dt_w} - D[C_2 + \delta(S - C + R)]\frac{t_2}{1 + \delta t_2}\frac{dt_1}{dt_w} \\ &= -\frac{D[C_2 + \delta(S - C + R)](t_1 + t_2)}{(1 + \delta t_2)^2}\frac{dt_2}{dt_w} < 0. \end{aligned}$$

Therefore, $G(t_w)$ is a strictly decreasing function of $t_w \in [0, \hat{t}_w)$. Furthermore, we have

$$\begin{aligned} \lim_{t_w \to \hat{t}_w^-} G(t_w) = {} & A + \frac{C_{11} + \alpha C}{\alpha}\left[W - \frac{D}{\alpha}\ln\left(1 + \frac{\alpha W e^{-\alpha \hat{t}_w}}{D}\right)\right] + \frac{D(C_{12} + \beta C)}{\beta^2}(e^{\beta \hat{t}_w} - \beta \hat{t}_w - 1) \\ & - \frac{D[C_2 + \delta(S - C + R)]}{\delta}\hat{t}_1 + \frac{D[C_2 + \delta(S - C + R)]}{\delta^2} - \frac{D[C_2 + \delta(S - C + R)]}{\delta^2}\lim_{t_w \to \hat{t}_w^-}\ln(1 + \delta t_2), \end{aligned}$$

where $\hat{t}_1 = \hat{t}_\mathrm{w} + \frac{1}{\alpha}\ln\left(1 + \frac{\alpha W e^{-\alpha \hat{t}_\mathrm{w}}}{D}\right)$. From Eq. (26), we obtain $t_2 \to \infty$ as $t_\mathrm{w} \to \hat{t}_\mathrm{w}^-$, hence it is easy to see that $\lim_{t_\mathrm{w}\to\hat{t}_\mathrm{w}^-} G_1(t_\mathrm{w}) = -\infty$. And,

$$\begin{aligned}G(0) &= A + \left(C + \frac{C_{11}}{\alpha}\right)(W - Dt_1) + \frac{D[C_2 + \delta(S - C + R)]}{\delta^2}[\delta t_2 - \ln(1 + \delta t_2)]\\ &\quad - D[C_2 + \delta(S - C + R)]\frac{(t_1 + t_2)t_2}{1 + \delta t_2}\\ &= A + \frac{W(C_{11} + \alpha C)}{\delta} - \frac{D(C_{11} + \alpha C)}{\alpha^2}\left[\left(1 + \frac{\alpha W}{D}\right)\ln\left(1 + \frac{\alpha W}{D}\right) - \frac{\alpha W}{D}\right]\\ &\quad - \frac{D[C_2 + \delta(S - C + R)]}{\delta^2}\ln\left\{\frac{D[C_2 + \delta(S - C + R)]}{D[C_2 + \delta(S - C + R)] - \delta W(C_{11} + \alpha C)}\right\}.\end{aligned}$$

Note that the value in the brace is well-defined because we have $\frac{C_2+\delta(S-C+R)}{\delta} > \frac{W(C_{11}+\alpha C)}{D}$. If we let $\Delta = G(0)$, i.e.,

$$\begin{aligned}\Delta &= A + \frac{W(C_{11} + \alpha C)}{\delta} - \frac{D(C_{11} + \alpha C)}{\alpha^2}\left[\left(1 + \frac{\alpha W}{D}\right)\ln\left(1 + \frac{\alpha W}{D}\right) - \frac{\alpha W}{D}\right]\\ &\quad - \frac{D[C_2 + \delta(S - C + R)]}{\delta^2}\ln\left\{\frac{D[C_2 + \delta(S - C + R)]}{D[C_2 + \delta(S - C + R)] - \delta W(C_{11} + \alpha C)}\right\},\end{aligned} \tag{32}$$

then we have the following result.

Lemma 2. *For* $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, *we have*:

(a) *If* $\Delta > 0$, *then the solution* $t_\mathrm{w}^* \in (0, \hat{t}_\mathrm{w})$ *which satisfies Eq.* (20) *not only exists but also is unique.*
(b) *If* $\Delta \leqslant 0$, *then the optimal value of* t_w *is* $t_\mathrm{w}^* = 0$.

Proof

(a) If $\Delta > 0$, i.e., $G(0) > 0$. Since $G(t_\mathrm{w})$ is a strictly decreasing function in $t_\mathrm{w} \in [0, \hat{t}_\mathrm{w})$, and $\lim_{t_\mathrm{w}\to\hat{t}_\mathrm{w}^-} G(t_\mathrm{w}) < 0$, by using the Intermediate Value Theorem, there exists a unique solution $t_\mathrm{w}^* \in (0, \hat{t}_\mathrm{w})$ such that $G(t_\mathrm{w}^*) = 0$.
(b) If $\Delta < 0$, i.e., $G(0) < 0$. Hence, for $t_\mathrm{w} \in [0, \hat{t}_\mathrm{w})$, we know the solution of $G(t_\mathrm{w}) = 0$ does not exist. For this situation, from Eqs. (16) and (31), we then obtain that $\frac{\partial P(t_\mathrm{w},t_2)}{\partial t_2} = \frac{G(t_\mathrm{w})}{(t_1+t_2)^2} < \frac{G(0)}{(t_1+t_2)^2} < 0$, which implies that a smaller value of t_2 causes a higher value of $P(t_\mathrm{w}, t_2)$. By using the finding of Eq. (27), we know that t_2 is a strictly increasing function of t_w; therefore, the maximum value of $P(t_\mathrm{w}, t_2)$ occurs at the boundary point $t_\mathrm{w}^* = 0$.

For the another case: $\Delta = 0$, i.e., $G(0) = 0$, then from the property that $G(t_\mathrm{w})$ is a strictly decreasing of function of $t_\mathrm{w} \in [0, \hat{t}_\mathrm{w})$, we see that $t_\mathrm{w}^* = 0$ is the unique solution. This completes the proof. □

When $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, Lemma 2(a) shows that $\Delta > 0$ is the condition for the existence and uniqueness of the solution. On the other hand, even if $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, Lemma 2(b) reveals that if the ordering cost, A, or the unit inventory cost per unit in OW, $C_{11} + \alpha C$, is relatively low so that $\Delta \leqslant 0$, the inventory model return to the one-warehouse problem.

The unique solution in Lemma 2(a) will be proved to be indeed a global maximum by checking the second order optimality conditions, that is, we have the following main result.

Theorem 2. *For* $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, *if* $\Delta > 0$, *then the point* (t_w^*, t_2^*) *which satisfies the Eqs.* (19) *and* (20) *simultaneously is the global maximum of the total profit per unit time.*

Proof. If $\Delta > 0$, then from Lemma 2(a), the solution $t_\mathrm{w}^* \in (0, \hat{t}_\mathrm{w})$ which satisfies Eq. (20) not only exists but also is unique. Hence, the value t_2^* can be determined by Eq. (29). Furthermore, since $0 < \frac{\mathrm{d}t_1}{\mathrm{d}t_\mathrm{w}} < 1$ and $\frac{\mathrm{d}K(t_\mathrm{w})}{\mathrm{d}t_\mathrm{w}} > 0$, we then obtain

$$\left.\frac{\partial^2 P(t_w,t_2)}{\partial t_w^2}\right|_{(t_w,t_2)=(t_w^*,t_2^*)} = \frac{-D}{t_1+t_2}\frac{dt_1}{dt_w}\left.\frac{dK(t_w)}{dt_w}\right|_{(t_w^*,t_2^*)} < 0,$$

$$\left.\frac{\partial^2 P(t_w,t_2)}{\partial t_2^2}\right|_{(t_w,t_2)=(t_w^*,t_2^*)} = -\left.\frac{D[C_2+\delta(S-C+R)]}{(t_1+t_2)(1+\delta t_2)^2}\right|_{(t_w^*,t_2^*)} < 0$$

and

$$\left.\frac{\partial^2 P(t_w,t_2)}{\partial t_w \partial t_2}\right|_{(t_w,t_2)=(t_w^*,t_2^*)} = 0.$$

Thus, the determinant of the Hessian matrix at the stationary point (t_w^*, t_2^*) is

$$\mathbf{H} = \left.\frac{\partial^2 P(t_w,t_2)}{\partial t_w^2}\right|_{(t_w^*,t_2^*)} \times \left.\frac{\partial^2 P(t_w,t_2)}{\partial t_2^2}\right|_{(t_w^*,t_2^*)} - \left.\left[\frac{\partial^2 P(t_w,t_2)}{\partial t_w \partial t_2}\right]^2\right|_{(t_w^*,t_2^*)} = \frac{D^2[C_2+\delta(S-C+R)]}{(t_1+t_2)^2(1+\delta t_2)^2}\frac{dt_1}{dt_w}\left.\frac{dK_1(t_w)}{dt_w}\right|_{(t_w^*,t_2^*)} > 0.$$

As a result, we can conclude that the stationary point (t_w^*, t_2^*) for our optimization problem is a global maximum. This completes the proof. □

Once the optimal solution (t_w^*, t_2^*) is obtained, we substitute (t_w^*, t_2^*) into Eqs. (12) and (14), the optimal ordering quantity per cycle and the maximum total profit per unit time $P(t_w^*, t_2^*)$ are as follows:

$$Q^* = W + \frac{D(e^{\beta t_w^*}-1)}{\beta} + \frac{D\ln(1+\delta t_2^*)}{\delta}$$

and

$$P(t_w^*, t_2^*) = D(S-C) - \frac{D[C_2+\delta(S-C+R)]t_2^*}{1+\delta t_2^*}. \tag{33}$$

4. Inventory problem without capacity constraint in OW

When the OW is so abundant that the RW is not used, the previous model reduces to the one-warehouse inventory problem. We remove the capacity constraint of the OW, and hence the total profit per unit time in Eq. (14) becomes

$$\Pi(t_1,t_2) = D(S-C) - \frac{A}{t_1+t_2} - \frac{C_{11}+\alpha C}{\alpha(t_1+t_2)}\left[\frac{D}{\alpha}(e^{\alpha t_1}-1) - Dt_1\right] - \frac{D[C_2+\delta(S-C+R)]}{\delta^2(t_1+t_2)}[\delta t_2 - \ln(1+\delta t_2)]. \tag{34}$$

Solving the necessary conditions: $\frac{\partial \Pi(t_1,t_2)}{\partial t_1} = 0$ and $\frac{\partial \Pi(t_1,t_2)}{\partial t_2} = 0$ for the maximum value of $\Pi\ (t_1, t_2)$, we get

$$\frac{[C_2+\delta(S-C+R)]t_2}{1+\delta t_2} - \frac{(C_{11}+\alpha C)(e^{\alpha t_1}-1)}{\alpha} = 0 \tag{35}$$

and

$$\begin{aligned} &A + \frac{C_{11}+\alpha C}{\alpha}\left[\frac{D}{\alpha}(e^{\alpha t_1}-1) - Dt_1\right] + \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)] \\ &\quad - D[C_2+\delta(S-C+R)]\frac{(t_1+t_2)t_2}{1+\delta t_2} = 0. \end{aligned} \tag{36}$$

After some algebraic manipulation, Eq. (35) can be rewritten as

$$t_2 = \frac{(C_{11}+\alpha C)(e^{\alpha t_1}-1)}{\alpha[C_2+\delta(S-C+R)] - \delta(C_{11}+\alpha C)(e^{\alpha t_1}-1)}. \tag{37}$$

Note that t_2 is a function of t_1, and when $t_1 \in \left(0, \frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}\right)$, we have $t_2 > 0$. Induced by Eq. (36), we define a function, $X(t_1)$, as follows:

$$X(t_1) = A + \frac{C_{11}+\alpha C}{\alpha}\left[\frac{D}{\alpha}(\mathrm{e}^{\alpha t_1}-1) - Dt_1\right] + \frac{D[C_2+\delta(S-C+R)]}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)] - D[C_2+\delta(S-C+R)]\frac{(t_1+t_2)t_2}{1+\delta t_2} \tag{38}$$

for $t_1 \in \left(0, \frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}\right)$ and t_2 is given as Eq. (37). By using the similar arguments as the above section, we can easily obtain the following two results. The proofs are omitted.

Lemma 3. *The point* $t_1^{**} \in \left(0, \frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}\right)$ *which satisfies the equation* $X(t_1) = 0$ *in* (38) *not only exists but also is unique.*

Theorem 3. *The point* (t_1^{**}, t_2^{**}) *which satisfies the Eqs. (35) and (36) simultaneously is the global maximum of the total profit per unit time* $\Pi(t_1, t_2)$.

From Theorem 3, once the optimal solution (t_1^{**}, t_2^{**}) is obtained, the optimal ordering quantity per cycle (denoted by Q^{**}), the maximum inventory level per cycle (denoted by B^{**}) and the maximum total profit per unit time $\Pi(t_1^{**}, t_2^{**})$ are as follows:

$$Q^{**} = \frac{D}{\alpha}(\mathrm{e}^{\alpha t_1^{**}}-1) + \frac{D\ln(1+\delta t_2^{**})}{\delta},$$
$$B^{**} = \frac{D}{\alpha}(\mathrm{e}^{\alpha t_1^{**}}-1)$$

and

$$\Pi(t_1^{**}, t_2^{**}) = D(S-C) - \frac{D[C_2+\delta(S-C+R)]t_2^{**}}{1+\delta t_2^{**}}. \tag{39}$$

Without the capacity constraint, we know that all of the ordering quantity can be stored in the OW. Under this situation, we want to compare the magnitude of the maximum inventory level B^{**} with the value W. Let us consider the following two cases: Case 1. $W \geqslant \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$ and Case 2. $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$. For Case 1: $W \geqslant \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, from Lemma 3, we know that $t_1^{**} \in \left(0, \frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}\right)$. Consequently, the maximum inventory level per cycle

$$B^{**} = \frac{D}{\alpha}(\mathrm{e}^{\alpha t_1^{**}}-1) < \frac{D}{\alpha}\left(\mathrm{e}^{\alpha\times\frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}}-1\right) \leqslant \frac{D}{\alpha}\left[\mathrm{e}^{\alpha\times\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)}-1\right] = W.$$

For Case 2: $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, we have

$$\frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\} > \frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right).$$

From Eqs. (32) and (38), it is not difficult to check that $X\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)\right) = \Delta$. Besides, it can be shown that $X(t_1)$ in Eq. (38) is a strictly decreasing function of $t_1 \in \left(0, \frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}\right)$ in conjunction with

$$\lim_{t_1\to 0^+} X(t_1) = A > 0$$

and

$$\lim_{t_1\to\frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}^-} X(t_1) < 0.$$

Now, we investigate the condition under which $X\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)\right) \leqslant 0$ or >0, and the following two cases arise.

(a) If $\Delta \leqslant 0$, then $X\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)\right)=\Delta \leqslant 0$. By the property of the function $X(t_1)$ and the Intermediate Value Theorem, we know that the optimal t_1^{**} must belong to the interval $\left(0,\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)\right]$. It in turn implies that the maximum inventory level per cycle

$$B^{**}=\frac{D}{\alpha}(e^{\alpha t_1^{**}}-1)\leqslant \frac{D}{\alpha}\left[e^{\alpha\times\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)}-1\right]=W.$$

(b) If $\Delta>0$, then $X\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)\right)=\Delta>0$. We know that the optimal t_1^{**} belongs to the interval $\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right),\frac{1}{\alpha}\ln\left\{1+\frac{\alpha[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}\right\}\right)$. It implies that the maximum inventory level per cycle

$$B^{**}=\frac{D}{\alpha}(e^{\alpha t_1^{**}}-1)>\frac{D}{\alpha}\left[e^{\alpha\times\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right)}-1\right]=W.$$

From the above arguments, we know that when the capacity of the OW is unrestricted (i.e., one-warehouse inventory problem), if $W\geqslant\frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, or $W<\frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$ and $\Delta\leqslant 0$, then the maximum ordering inventory quantity per cycle B^{**} is less or equal to W, i.e., $B^{**}\leqslant W$. On the other hand, if $W<\frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$ and $\Delta>0$, then the maximum ordering quantity per cycle B^{**} is larger than W, i.e., $B^{**}>W$.

Next, we want to compare the difference between the total profit per unit time of the one-warehouse inventory problem with the two-warehouse inventory problem which the RW is not required. It can be shown that $\Pi\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right),t_2^*\right)=P(0,t_2^*)$, where t_2^* is the root which satisfies Eq. (23). However, since (t_1^{**},t_2^{**}) is the optimal solution such that $\Pi\ (t_1,t_2)$ is maximum, hence we have that $\Pi\left(\frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D}\right),t_2^*\right)<\Pi(t_1^{**},t_2^{**})$, which implies $P(0,t_2^*)<\Pi(t_1^{**},t_2^{**})$. As a result, once the condition of Theorem 1 or Lemma 2(b) is satisfied, since $P(0,t_2^*)<\Pi(t_1^{**},t_2^{**})$, the inventory model with two warehouses will return to the one-warehouse inventory problem such that Theorem 3 applies.

5. Some special cases

In this section, the two-warehouse inventory model is illustrated for some special cases. We construct them as follows:

Case 1. Without shortage

When $\delta\to\infty$ (i.e., the fraction of shortages backordered is zero), from Eq. (29), we get $t_2\approx 0$. The model reduce to the case where shortages are not allowed and the total profit per unit time in Eq. (14) approaches to

$$\begin{aligned}P_1(t_w)&\equiv P(t_w,0)\\&=D(S-C)-\frac{A}{t_1}-\frac{C}{t_1}\left[W+\frac{D}{\beta}(e^{\beta t_w}-1)-Dt_1\right]-\frac{C_{11}}{\alpha t_1}[W-D(t_1-t_w)]-\frac{DC_{12}}{\beta^2 t_1}(e^{\beta t_w}-\beta t_w-1),\end{aligned}\tag{40}$$

where t_1 is a function of t_w and be defined as in Eq. (8). The necessary condition to find the optimal solution of $P_1(t_w)$ is

$$\begin{aligned}\frac{dP_1(t_w)}{dt_w}&=\frac{A}{t_1^2}\frac{dt_1}{dt_w}+\frac{C}{t_1^2}\left[W+\frac{D}{\beta}(e^{\beta t_w}-1)-Dt_1\right]\frac{dt_1}{dt_w}-\frac{DC}{t_1}\left(e^{\beta t_w}-\frac{dt_1}{dt_w}\right)+\frac{C_{11}}{\alpha t_1^2}[W-D(t_1-t_w)]\frac{dt_1}{dt_w}\\&\quad+\frac{DC_{11}}{\alpha t_1}\left(\frac{dt_1}{dt_w}-1\right)+\frac{DC_{12}}{\beta^2 t_1^2}(e^{\beta t_w}-\beta t_w-1)\frac{dt_1}{dt_w}-\frac{DC_{12}}{\beta t_1}(e^{\beta t_w}-1)\\&=\frac{1}{t_1^2(1+\alpha W e^{-\alpha t_w}/D)}\left\{A+\frac{W(C_{11}+\alpha C)}{\alpha}-\frac{D(C_{11}+\alpha C)(t_1-t_w)}{\alpha}+\frac{D(C_{12}+\beta C)(e^{\beta t_w}-\beta t_w-1)}{\beta^2}\right.\\&\quad\left.-D(C_{11}+\alpha C)\frac{We^{-\alpha t_w}}{D}t_1-\frac{D(C_{12}+\beta C)(e^{\beta t_w}-1)}{\beta}\left(1+\frac{\alpha W e^{-\alpha t_w}}{D}\right)t_1\right\}\\&=\frac{1}{t_1^2(1+\alpha W e^{-\alpha t_w}/D)}\left\{A+\frac{W(C_{11}+\alpha C)}{\alpha}-\frac{D(C_{11}+\alpha C)(t_1-t_w)}{\alpha}\right.\\&\quad\left.+\frac{D(C_{12}+\beta C)(e^{\beta t_w}-\beta t_w-1)}{\beta^2}-Dt_1K(t_w)\right\}=0,\end{aligned}\tag{41}$$

which implies

$$A+\frac{W(C_{11}+\alpha C)}{\alpha}-\frac{D(C_{11}+\alpha C)(t_1-t_{\mathrm{w}})}{\alpha}+\frac{D(C_{12}+\beta C)(\mathrm{e}^{\beta t_{\mathrm{w}}}-\beta t_{\mathrm{w}}-1)}{\beta^2}-Dt_1K(t_{\mathrm{w}})=0.$$

Let

$$L(t_{\mathrm{w}})=A+\frac{W(C_{11}+\alpha C)}{\alpha}+\frac{D(C_{12}+\beta C)(\mathrm{e}^{\beta t_{\mathrm{w}}}-\beta t_{\mathrm{w}}-1)}{\beta^2}-\frac{D(C_{11}+\alpha C)(t_1-t_{\mathrm{w}})}{\alpha}-Dt_1K(t_{\mathrm{w}}),t_{\mathrm{w}}\geqslant 0. \tag{42}$$

By using the analogous derivations as above section, we can show $L(t_{\mathrm{w}})$ is a strictly decreasing function in $t_{\mathrm{w}}\in[0,\infty)$, and hence there exists a unique value $t_{\mathrm{w}}^{\Delta}\in(0,\infty)$ such that $L(t_{\mathrm{w}}^{\Delta})=0$. Substituting t_{w}^{Δ} into Eq. (8), a corresponding t_1^{Δ} can be determined, and thus the optimal quantity per cycle (denoted by Q^{Δ}) and the maximum total profit per unit time $P_1(t_{\mathrm{w}}^{\Delta})$ are as follows:

$$Q^{\Delta}=W+\frac{D(\mathrm{e}^{\beta t_{\mathrm{w}}^{\Delta}}-1)}{\beta}$$

and

$$P_1(t_{\mathrm{w}}^{\Delta})=D(S-C)-DK(t_{\mathrm{w}}^{\Delta}). \tag{43}$$

Case 2. Without stock

We consider the case that retailers do not carry any stock on hand and just accept backorders. In this situation, the inventory model starts with shortages, and keep the negative inventory level in the interval $(0,t_2)$. Hence, the total profit per unit time in Eq. (34) becomes

$$P_2(t_2)\equiv P(0,t_2)=D(S-C)-\frac{A}{t_2}-\frac{D[C_2+\delta(S-C+R)]}{\delta^2 t_2}[\delta t_2-\ln(1+\delta t_2)] \tag{44}$$

The necessary condition to find the optimal solution of $P_2(t_2)$ is

$$\frac{\mathrm{d}P_2(t_2)}{\mathrm{d}t_2}=\frac{1}{t_2^2}\left\{A+\frac{D[C_2+\delta(S-C+R)]}{\delta^2}\left[\frac{\delta t_2}{1+\delta t_2}-\ln(1+\delta t_2)\right]\right\}=0,$$

which implies

$$A+\frac{D[C_2+\delta(S-C+R)]}{\delta^2}\left[\frac{\delta t_2}{1+\delta t_2}-\ln(1+\delta t_2)\right]=0. \tag{45}$$

We define a new function as follows:

$$M(t_2)=A+\frac{D[C_2+\delta(S-C+R)]}{\delta^2}\left[\frac{\delta t_2}{1+\delta t_2}-\ln(1+\delta t_2)\right],\quad t_2\geqslant 0. \tag{46}$$

For any given $t_2\in[0,\infty)$, because $\frac{\mathrm{d}M(t_2)}{\mathrm{d}t_2}=-D[C_2+\delta(S-C+R)]\frac{t_2}{(1+t_2)^2}<0$, $M(0)=A$, and $\lim_{t_2\to\infty}M(t_2)=-\infty$. Hence, there exists a unique value $t_2^{\#}\in(0,\infty)$ such that $M(t_2^{\#})=0$. Thus the maximum total profit per unit time

$$P_2(t_2^{\#})=D(S-C)-\frac{D[C_2+\delta(S-C+R)]t_2^{\#}}{1+\delta t_2^{\#}} \tag{47}$$

follows.

Next, for the two-warehouse inventory model with partial backlogging discussed originally and the two special cases, we will demonstrate which of these three cases is profitable. Due to the relations shown in Eqs. (42) and (46), Eq. (20) can be written as

$$A=L(t_{\mathrm{w}})+M(t_2). \tag{48}$$

Note that Eq. (48) is the necessary condition to find the optimal solution of $P(t_{\mathrm{w}},t_2)$. Then we have the following result.

Theorem 4. *For* $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, *if* $\Delta > 0$, *then* $P(t_w^*, t_2^*) > \max\{P_1(t_w^\Delta), P_2(t_2^\#)\}$.

Proof. Because t_w^Δ and (t_w^*, t_2^*) are the optimal solutions of $P_1(t_w)$ in Eq. (40) and $P(t_w, t_2)$ in Eq. (14) respectively, from Eqs. (42) and (48), we have

$$L(t_w^\Delta) = 0 \tag{49}$$

and

$$A = L(t_w^*) + M(t_2^*). \tag{50}$$

Eq. (50) can be rewritten as

$$L(t_w^*) = A - M(t_2^*) > A - M(0) = 0, \tag{51}$$

because $M(t_2)$ is a strictly decreasing function and $M(0) = A$. Comparing Eqs. (49) and (51), we get

$$L(t_w^*) > L(t_w^\Delta). \tag{52}$$

Recall that $L(t_w)$ is a strictly decreasing function in $t_w \in [0, \infty)$, Eq. (52) implies $t_w^\Delta > t_w^*$. Then, from Eqs. (33) and (43), we obtain

$$\begin{aligned} P(t_w^*, t_2^*) &= D(S-C) - \frac{D[C_2+\delta(S-C+R)]t_2^*}{1+\delta t_2^*} = D(S-C) - DK(t_w^*) > D(S-C) - DK(t_w^\Delta) \\ &= P_1(t_w^\Delta). \end{aligned} \tag{53}$$

Similarly, we can get $t_2^\# > t_2^*$. Then, from Eqs. (33) and (47), we obtain

$$P(t_w^*, t_2^*) = D(S-C) - \frac{D[C_2+\delta(S-C+R)]t_2^*}{1+\delta t_2^*} > D(S-C) - \frac{D[C_2+\delta(S-C+R)]t_2^\#}{1+\delta t_2^\#} = P_2(t_2^\#). \tag{54}$$

Combining Eqs. (53) and (54), we get

$$P(t_w^*, t_2^*) > \max\{P_1(t_w^\Delta), P_2(t_2^\#)\}.$$

This completes the proof. □

Furthermore, let $\Pi_1(t_1)$ represent the total profit per unit time in the one-warehouse problem without shortages and let $t_1^{\Delta\Delta}$ denote the optimal solution of $\Pi_1(t_1)$; and let $\Pi_2(t_2)$ represent the total profit per unit time in the one-warehouse problem without stock and let $t_2^\#$ denote the optimal solution of $\Pi_2(t_2)$. By using the analogous derivations as in Theorem 4, we can easily obtain the following result. The proof is omitted.

Theorem 5. *For the one-warehouse problem*, $\Pi(t_1^{**}, t_2^{**}) > \max\{\Pi_1(t_1^{\Delta\Delta}), \Pi_2(t_2^\#)\}$.

Up to now, we present three inventory policies: without shortage, without stock and partial backlogging. From Theorems 4 and 5, we show that the inventory policy with partial backlogging is profitable.

6. Numerical example

In this section, our illustration begins from a two-warehouse inventory problem under the condition $W < \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$ with a precise judgment criterion, Δ. Because t_w^*, t_1^*, t_2^* and T^* cannot be determined in the closed forms, they have to be solved numerically by using some computer algorithm. While $\Delta > 0$, Theorem 2 applies and we can obtain the value of t_w^* from Eq. (20) by using Newton–Raphson Method (or any bisection method). Once the optimal t_w^* has been determined, the optimal t_1^*, t_2^* and T^* follows by using Eqs. (28)–(30), respectively. On the other hand, while $\Delta \leqslant 0$, Theorem 3 applies and the optimal solution can be obtained by using the similar technique.

In order to illustrate the proposed model, we provide some computational results for a numerical example with the parameters specified in the following: $D = 1000$, $A = 100$, $C = 10$, $C_{11} = 0.2$, $C_{12} = 0.5$, $C_2 = 2$, $S = 15$, $\alpha = 0.02$, $\beta = 0.05$, $R = 7$ and $\delta \in \{0.25, 0.5, 1, 2.5, 5, \infty\}$ with suitable units. Note that $\delta \to \infty$ implies that shortages are not allowed. Besides, we consider the problem for $W = \{300, 500, 700\}$ to examine the practical inventory system. In our illustration, we denote $\Phi = \frac{D[C_2+\delta(S-C+R)]}{\delta(C_{11}+\alpha C)}$, and have $\Phi \in \{50{,}000, 40{,}000, 35{,}000, 32{,}000, 31{,}000, 30{,}000\}$ corresponding to different given δ. Furthermore, by calculating Δ in Eq. (32), we can check the RW is required or not. When $W = 300$ and 500, we have $\Delta > 0$ for all δ such that Theorem 2 applies. Next, when $W = 700$, we have $\Delta < 0$ for $\delta \in \{0.25, 0.5, 1\}$ such that Theorem 3 applies, and $\Delta > 0$ for $\delta \in \{2.5, 5, \infty\}$ such that Theorem 2 applies. The computational results for the optimal value of t_w^*, t_1^*, T^*, t_1^*/T^* (the optimal service rate), $P(t_w^*, t_2^*)$, $\Pi(t_1^{**}, t_2^{**})$ and B^* (the maximum inventory level) with respect to different values of W and δ are shown in Table 1. For comparison, the result of the special case, $\delta = 0$ (i.e. complete backlogging) is also listed in the same table. The results were obtained using Mathematica version 4.0.

Based on Table 1, the effects of W and δ on the maximum total profit per unit time, ordering quantity and maximum inventory level are portrayed in Figs. 2–4, respectively. Besides, the following inferences can be made from the results in Table 1 and Figs. 2–4.

1. For fixed W, an increase in the value of δ (which decreases the backlogging rate) will result in a decrease in T^*, Q^* and $P(t_w^*, t_2^*)$, but an increase in t_w^*, t_1^*, t_1^*/T^* and B^*.
2. For fixed W, the maximum profit occurs at $\delta = 0$ (i.e. complete backlogging), and the minimum profit occurs at $\delta \to \infty$ (i.e. without shortage).

Table 1
Effects of W and δ on the optimal solution

		Complete backlogging ($\delta = 0$)	δ					Without shortage ($\delta \to \infty$)
			0.25	0.5	1	2.5	5	
$W = 300$	Φ	∞	50,000	40,000	35,000	32,000	31,000	30,000
	Δ	78.44	80.59	81.13	81.52	81.81	81.92	82.04
	t_w^*	0.1620	0.1842	0.1909	0.1959	0.1999	0.2015	0.2032
	t_1^*	0.4601	0.4822	0.4888	0.4939	0.4979	0.4994	0.5011
	T^*	0.6017	0.5443	0.5287	0.5171	0.5082	0.5048	0.5011
	t_1^*/T^*	0.7647	0.8859	0.9246	0.9551	0.9797	0.9894	1.0000
	Q^*	604.26	546.70	531.22	519.84	511.12	507.80	504.22
	B^*	462.64	485.08	491.78	496.90	500.94	502.51	504.22
	$P(t_w^*, t_2^*)$	4716.77	4694.25	4687.54	4682.40	4678.34	4676.76	4675.04
$W = 500$	Δ	40.17	46.14	47.65	48.72	49.53	49.84	50.17
	t_w^*	0.0619	0.0783	0.0830	0.0866	0.0894	0.0905	0.0916
	t_1^*	0.5588	0.5750	0.5797	0.5833	0.5860	0.5871	0.5883
	T^*	0.6900	0.6316	0.6158	0.6042	0.5953	0.5919	0.5883
	t_1^*/T^*	0.8099	0.9104	0.9414	0.9653	0.9844	0.9919	1.0000
	Q^*	693.21	634.60	618.96	607.51	598.76	595.43	591.85
	B^*	562.02	578.43	583.19	586.78	589.59	590.68	591.85
	$P(t_w^*, t_2^*)$	4737.61	4721.1	4716.32	4712.7	4709.87	4708.78	4707.60
$W = 700$	Δ	−17.15	−5.46	−2.50	−0.38	1.21	1.81	2.45
	t_1^{**}	0.6425	0.6770	0.6866	0.6938	–	–	–
	T^{**}	0.7718	0.7323	0.7218	0.7142	–	–	–
	t_1^{**}/T^{**}	0.8324	0.9245	0.9513	0.9715	–	–	–
	Q^{**}	775.98	736.51	726.25	718.82	–	–	–
	B^{**}	646.65	681.61	691.38	698.66	–	–	–
	$\Pi(t_1^{**}, t_2^{**})$	4741.34	4727.36	4723.45	4720.54	–	–	–
	t_w^*	–	–	–	–	0.0017	0.0026	0.0035
	t_1^*	–	–	–	–	0.6968	0.6977	0.6986
	T^*	–	–	–	–	0.7058	0.7023	0.6986
	t_1^*/T^*	–	–	–	–	0.9872	0.9934	1.0000
	Q^*	–	–	–	–	710.61	707.17	703.49
	B^*	–	–	–	–	701.70	702.56	703.49
	$P(t_w^*, t_2^*)$	–	–	–	–	4718.28	4717.41	4716.48

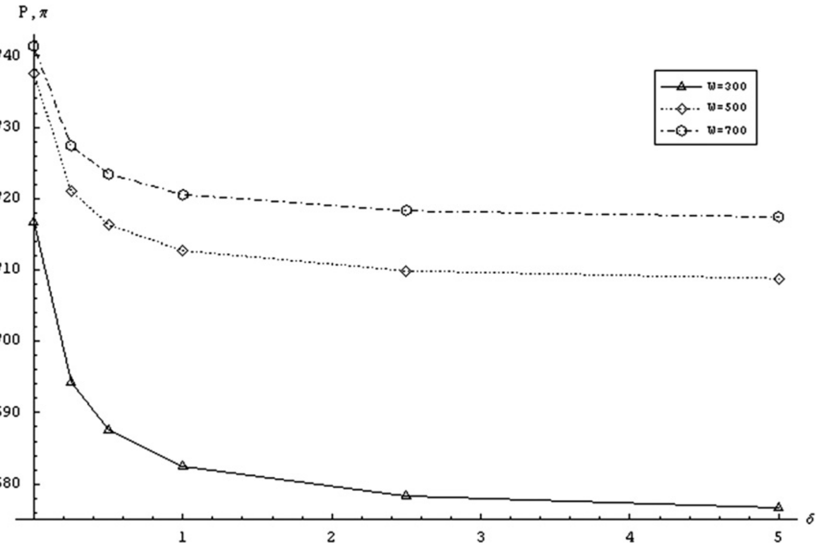

Fig. 2. Effects of δ and W on the maximum total profit per unit time.

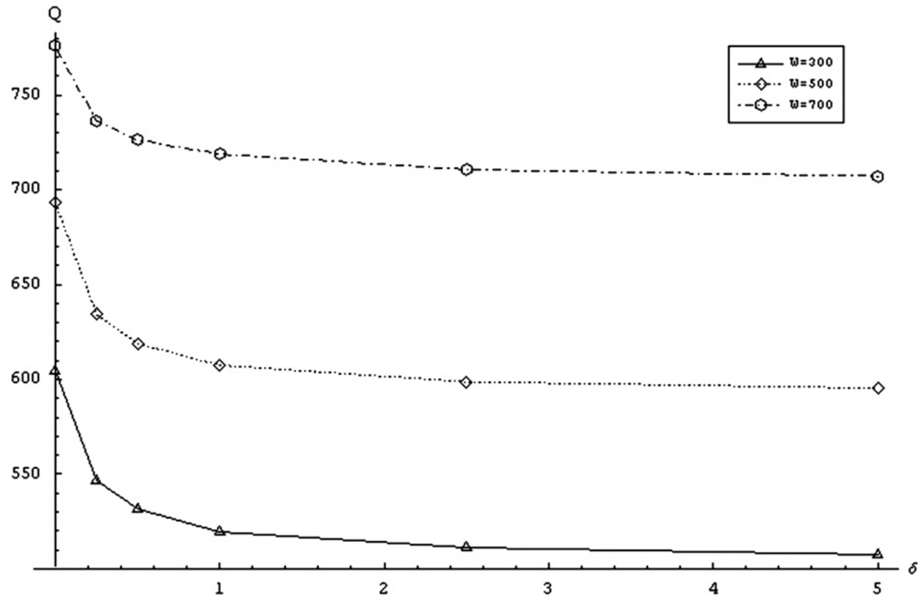

Fig. 3. Effects of δ and W on ordering quantity (Q).

3. For fixed δ, a higher value of W results in higher values for t_1^*, T^*, t_1^*/T^*, Q^*, B^* and $P(t_w^*, t_2^*)$, but a lower value for t_w^*.
4. Q^*, B^* and $P(t_w^*, t_2^*)$ are less sensitive to δ when it's value is larger.

The inferences above are consistent with the intuitive reasoning. For a fixed W, as δ decreases, the backlogging rate will increase and result in a larger profit. Hence, in order to increase the profit per unit time, the retailer should reduce the value of the backlogging parameter δ. When δ equals to zero, the model reduces to the case of complete backlogging, and has the maximum profit per unit time. On the other hand, for fixed δ, as W increases, the retailer should increase the ordering quantity and shorten the duration that inventory is stored in the RW.

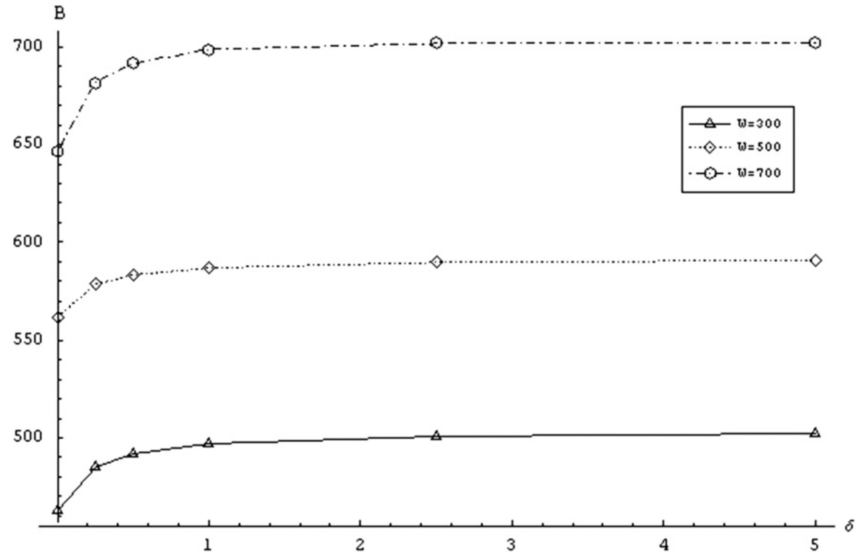

Fig. 4. Effects of δ and W on maximum inventory level (B).

7. Concluding remarks

In this paper, an inventory model is developed for deteriorating items with finite warehouse capacity, permitting shortage and time-proportional backlogging rate. Holding costs and deterioration costs are different in OW and RW due to different preservation environments. The inventory costs (including holding cost and deterioration cost) in RW are assumed to be higher than those in OW. To reduce the inventory costs, it will be economical for firms to store goods in OW before RW, but clear the stocks in RW before OW. In particular, the backlogging rate considered to be a decreasing function of the waiting time for the next replenishment is more realistic. In practice, we can observe periodically the proportion of demand which would accept backlogging and the corresponding waiting time for the next replenishment. Then the statistical techniques, such as the nonlinear regression method, can be used to estimate the backlogging rate. Furthermore, we show that the inventory policy with partial backlogging is more profitable than those without shortage and without stock. We also provide some useful properties for finding the optimal replenishment policy and show in a rigorous way that the policy suggested is indeed optimal. By using the presented approach, we can easily decide whether the retailer has to rent another warehouse and obtain the optimal replenishment policy among those cases with the help of the auxiliary values.

The proposed model can be extended in several ways. For instance, we may consider finite rate of replenishment. Also, we could extend the deterministic demand function to stochastic demand patterns. Furthermore, we could generalize the model to allow for permissible delay in payments.

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References

[1] R.V. Hartley, Operations Research—A Managerial Emphasis, Good Year Publishing Company, California, 1976, pp. 315–317.
[2] K.V.S. Sarma, A deterministic inventory model with two level of storage and an optimum release rule, Opsearch 20 (1983) 175–180.
[3] T.A. Murdeshwar, Y.S. Sathe, Some aspects of lot size model with two levels of storage, Opsearch 22 (1985) 255–262.
[4] U. Dave, On the EOQ models with two levels of storage, Opsearch 25 (1988) 190–196.

[5] K.V.S. Sarma, A deterministic order level inventory model for deteriorating items with two storage facilities, European Journal of Operational Research 29 (1987) 70–73.
[6] T.P.M. Pakkala, K.K. Achary, Discrete time inventory model for deteriorating items with two warehouses, Opsearch 29 (1992) 90–103.
[7] T.P.M. Pakkala, K.K. Achary, A deterministic inventory model for deteriorating items with two warehouses and finite replenishment rate, European Journal of Operational Research 57 (1992) 71–76.
[8] A. Goswami, K.S. Chaudhuri, An economic order quantity model for items with two levels of storage for a linear trend in demand, Journal of the Operational Research Society 43 (1992) 157–167.
[9] A. Goswami, K.S. Chaudhuri, On an inventory model with two levels of storage and stock-dependent demand rate, International Journal of Systems Sciences 29 (1998) 249–254.
[10] A.K. Bhunia, M. Maiti, A two warehouse inventory model for a linear trend in demand, Opsearch 31 (1994) 318–329.
[11] A.K. Bhunia, M. Maiti, A two-warehouse inventory model for deteriorating items with a linear trend in demand and shortages, Journal of the Operational Research Society 49 (1998) 287–292.
[12] L. Benkherouf, A deterministic order level inventory model for deteriorating items with two storage facilities, International Journal of Production Economics 48 (1997) 167–175.
[13] S. Kar, A.K. Bhunia, M. Maiti, Deterministic inventory model with two levels of storage, a linear trend in demand and a fixed time horizon, Computers & Operations Research 28 (2001) 1315–1331.
[14] H.L. Yang, Two-warehouse inventory models for deteriorating items with shortages under inflation, European Journal of Operational Research 157 (2004) 344–356.
[15] Y.W. Zhou, A multi-warehouse inventory model for items with time-varying demand and shortages, Computers & Operations Research 30 (2003) 2115–2134.
[16] P.L. Abad, Optimal pricing and lot sizing under conditions of perishability and partial backordering, Management Science 42 (1996) 1093–1104.
[17] P.L. Abad, Optimal price and order size for a reseller under partial backordering, Computers & Operations Research 28 (2001) 53–65.